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Strong Solutions of the Navier-Stokes Equation in Morrey Spaces

Tosio Kato

— Dedicated to Felix E. Browder

Abstract. It is shown that the nonstationary Navier-Stokes equation (NS) in $\mathbb{R}^+ \times \mathbb{R}^m$ is well posed in certain Morrey spaces $M_{p,\lambda}(\mathbb{R}^m)$ (see the text for the definition: in particular $M_{p,0} = L^p$ if $p > 1$ and $M_{1,0}$ is the space of finite measures), in the following sense. Given a vector $a \in M_{p,m-p}$ with div $a = 0$ and with certain supplementary conditions, there is a unique local (in time) solution (velocity field) $u(t, \cdot) \in M_{p,m-p}$, which is smooth for $t > 0$ and takes the initial value a at least in a weak sense. u is a global solution if a is sufficiently small. Of particular interest is the space $M_{1,m-1}$, which admits certain measures; thus a may be a surface measure on a smooth $(m-1)$- dimensional surface in $\mathbb{R}^m$. The regularity of solutions and the decay of global solutions are also considered. The associated vorticity equation (for the vorticity $\zeta = \partial \wedge u$) can similarly be solved in (tensor-valued) $M_{1,m-2}$, which is also a space of measures of another kind.

Introduction

In this paper we continue the study of *strong solutions* of the Cauchy problem for the free Navier-Stokes equation in $\mathbb{R}^m$ (cf. [5]):

$$(NS)\qquad \begin{aligned} &\partial_t u - \Delta u + \Pi\partial \cdot (u \otimes u) = 0, \\ &\partial \cdot u = 0, \quad u(0) = a \quad (\partial \cdot a = 0). \end{aligned}$$

Here $\partial = (\partial_1, \dots, \partial_m)$, $\partial_j = \partial/\partial x_j$, $u: [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ is the velocity field, $u \otimes u$ is the tensor with jk-components $u_j u_k$, $\partial \cdot (u \otimes u)$ is a vector with j-th component $\partial_k(u_k u_j) = (\partial_k u_k) u_j$ (with summation convention), and Π is the projection operator onto divergence free vectors along gradients. Π is an $m \times m$-matrix with elements $\delta_{jk} + R_j R_k$, where $R_j = \partial_j(-\Delta)^{-1/2}$ are singular integral operators (the Riesz transforms). The pressure field π may be recovered, if necessary, by $\partial\pi = (\Pi - 1)\partial \cdot (u \otimes u)$.

Roughly speaking, a strong solution $u(t)$ of (NS) is smooth for $t > 0$ but the initial velocity a may be nonsmooth and of slow spatial decay. It was shown

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in [5] that if $a \in L^m$, then a unique strong solution $u(t) \in L^m$ exists for short time, and for all time if $\|a\|_m$ is sufficiently small. (L^p means $L^p(\mathbb{R}^m)$ unless otherwise stated, and $\| \quad \|_p$ denotes the L^p-norm.) No other L^p is known to have this property.

Recently Giga and Miyakawa [3] proved that if the *initial vorticity* $\partial \wedge a$ is a Radon measure belonging to a certain *Morrey space*, then (NS) is uniquely solvable in a similar sense. The Morrey space $M_{p,\lambda}$ on $\mathbb{R}^m$ is defined, for $0 \leq \lambda < m$, as the class of (scalar or vector valued) functions f such that the integral of $|f|^p$ on a ball of radius R is dominated by const. R^λ for all $R > 0$, uniformly in the position of the ball. (By extension $M_{1,\lambda}$ includes some measures.) $M_{p,0}$ is identical with L^p (or the space $\mathcal{M}$ of finite measures if $p = 1$). For $0 < \lambda < m$, $M_{p,\lambda}$ may include various nonsmooth functions or measures that have no decay in some directions. (For example, $f(x) = \delta(x_1)\delta(x_2)$ on $\mathbb{R}^3$ belongs to $M_{1,1}$.)

The main assumption in [3] is that $\partial \wedge a$ belongs to (vector-valued) $M_{1,1}(\mathbb{R}^3)$. This assumption implies certain differentiability of a.

The purpose of the present paper is to solve (NS) for $u(t)$ in Morrey spaces, without considering the vorticity $\partial \wedge u$ except in the last section. A typical result is that if $1 < p \leq m$, (NS) is well posed in $M_{p,m-p}$ in the following sense: if $a \in M_{p,m-p}$ with small norm, then a global solution $u(t) \in M_{p,m-p}$ exists, and it is unique within certain restrictions. (For $p = m$, this reduces to the case of [5].) Since it is known that $\partial \wedge a \in M_{1,m-2}$ implies $a \in M_{p,m-p}$ for any p with $1 \leq p < m/(m-1)$ (see Lemma 4.1), this partially generalizes the result of [3] by eliminating the differentiability of a.

With some modifications, similar results hold for $p = 1$ as well as for local existence for large a. For details see Theorems I, II in sections 5,6. The case $p = 1$ is interesting in that *certain measures are allowed as the initial velocity* a. For example, a may be a divergence free, tangential vector measure concentrated on a smooth surface in $\mathbb{R}^3$ (see Remark 6.3).

For the global solution u we have the decay $\|u(t)\|_\infty = O(t^{-1/2})$. This rate is improved under the additional condition that $a \in M_{q,\mu}$ for another pair (q, μ). In particular $a \in L^1$ (or $\mathcal{M}$) will lead to the "maximal" decay $\|u(t)\|_r = O(t^{m/2r-m/2})$, $1 < r \leq \infty$. For details see Theorem III (section 7). The smoothness of the solution for $t > 0$ is proved in Theorem IV (section 8). Finally we consider the vorticity equation, and recover some other results from [3] (section 9).

The proofs of these theorems are basically the same as in [5], but more complicated due to the fact that neither translations nor heat operators form C_0-semigroups on $M_{p,\lambda}$ with $\lambda > 0$. For this reason we have to interpret the initial condition $u(0) = a$ in a weak sense, or else restrict a to certain subspaces of $M_{p,\lambda}$.

Section 1 summarizes the main properties of the Morrey spaces and their subspaces, and introduces a convenient geometric notation, by which $M_{p,\lambda}$ is represented by a point $A = (1/p, \alpha) \in \mathbb{R}^2$ with $\alpha = (m - \lambda)/p$. In sections 2 and 3, we study the behavior of the heat operator $U(t) = e^{t\Delta}$ and the translation group. In particular we identify the maximal subspace of $M_{p,\lambda}$ on which $U(t)$ has the C_0-property. It turns out that it is also the maximal subspace on which translations form a C_0-group. In section 4 we prove the boundedness of potential operators and singular integral operators, including the projection Π. In section 5 we solve the integral equation associated with (NS). Sections 6 to 9 contain the main results of this paper.

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1. Morrey spaces and their subspaces

We summarize the basic properties of Morrey spaces that we need in the sequel, with sketches of proof when necessary.

1. The Morrey space $M_{p,\lambda} = M_{p,\lambda}(\mathbb{R}^m)$ is defined as the subspace of $L^p_{\text{loc}}(\mathbb{R}^m)$ with the norm defined by

$$\begin{aligned} &\|f\|_{p,\lambda} = \sup\{R^{-\lambda/p}\|f\|_{p;x,R}\,; x \in \mathbb{R}^m, R > 0\} \\ &1 \le p < \infty,\ 0 \le \lambda < m, \end{aligned} \tag{1.1}$$

where $\|f\|_{p;x,R}$ denotes the L^p-norm of f on $B_R(x)$ (the closed ball in $\mathbb{R}^m$ with center x and radius R). If $p = 1$, we allow (signed) measures for $f \in M_{1,\lambda}$, with $\|f\|_{1;x,R}$ denoting the total variation of f on $B_R(x)$. $M_{p,\lambda}$ is a Banach space. For basic results on Morrey spaces, see e.g. Campanato [1], Peetre [6].

In particular $M_{p,0} = L^p$ for $p > 1$, and $M_{1,0}$ is the Banach space of finite measures, which we denote by $\mathcal{M}$. For various reasons we find it convenient to include L^∞ among the Morrey spaces, but *the indices in the notation $M_{p,\lambda}$ will always be restricted to* $1 \le p < \infty$, $0 \le \lambda < m$, notwithstanding that (1.1) makes sense for $\lambda = m$ and the resulting space is equivalent to L^∞ (irrespective of the value of p).

We define a subset $\dot{M}_{p,\lambda}$ of $M_{p,\lambda}$ by requiring, in addition to $\|f\|_{p,\lambda} < \infty$, that

$$R^{-\lambda/p} \sup\{\|f\|_{p;x,R}\,; x \in \mathbb{R}^m\} \to 0 \text{ as } R \to 0. \tag{1.1a}$$

Note that (1.1a) is trivially satisfied if $f \in L^\infty$.

We define another subset $\ddot{M}_{p,\lambda}$ of $M_{p,\lambda}$ by the condition that

$$\|\tau_\xi f - f\|_{p,\lambda} \to 0 \text{ as } \xi \to 0, \tag{1.1b}$$

where τ_ξ denotes translation: $\tau_\xi f(x) = f(x-\xi)$, x, $\xi \in \mathbb{R}^m$.

It follows from the Banach-Steinhaus theorem that $\dot{M}_{p,\lambda}$ and $\ddot{M}_{p,\lambda}$ are closed subspaces of $M_{p,\lambda}$. We have $M_{p,0} = \dot{M}_{p,0} = \ddot{M}_{p,0} = L^p$ for $p > 1$, by the uniform integrability of L^1-functions and the continuity of translations on L^p. Actually we have $\ddot{M}_{p,\lambda} \subset \dot{M}_{p,\lambda}$, as will be proved later (section 3).

2. $M_{p,\lambda}$ forms a two-parameter family, in which there is a one- parameter family of inclusion relations (cf. [6])

$$M_{q,\mu} \subset M_{p,\lambda} \quad \text{if} \quad (m-\lambda)/p = (m-\mu)/q, \quad p \le q. \tag{1.2}$$

To see this, it suffices to note that $\|f\|_{p;x,R} \le cR^{m/p-m/q}\,\|f\|_{q;x,R}$ by the Hölder inequality. The same proof applies to show that $\dot{M}_{p,\lambda}$ and $\ddot{M}_{p,\lambda}$ also satisfy the same inclusion relations.

Relation (1.2) suggests that it is convenient to introduce a new set of parameters to describe the Morrey spaces (cf. [6]). We shall write

$$M_{p,\lambda} = M(A), \quad A = (1/p, \alpha) \in \blacktriangle, \quad \alpha = (m-\lambda)/p > 0, \tag{1.3}$$

where $\blacktriangle = \blacktriangle_m$ denotes the right triangle with vertices $O = (0,0), (1,0), (m,0)$, with the bottom side excluded except for the origin O, with the convention that $M(O) = L^\infty$ (see Fig. 1). For $A = (1/p, \alpha)$, we write $1/p = x(A)$, $\alpha = y(A)$; $y(A)$ will be called the *height* of A. $y(A) = 0$ occurs only for $A = O$. The closed segment connecting two points A, B will be denoted by $[AB]$ or $[A,B]$. $]AB[$ denotes the open segment. (The parameter α is the same as in [6], with the sign reversed.)

Corresponding to (1.3), the norm $\|f\|_{p,\lambda}$ will be denoted by $\|f; M(A)\|$ or, more simply, by $\|f;A\|$ (with $\|f;O\| = \|f\|_\infty$). Thus

$$\begin{aligned} &\|f;A\| = \sup\{R^{-m/p+\alpha}\,\|f\|_{p;x,R}\,; x \in \mathbb{R}^m, R > 0\}, \\ &A = (1/p, \alpha) \in \blacktriangle. \end{aligned} \tag{1.4}$$

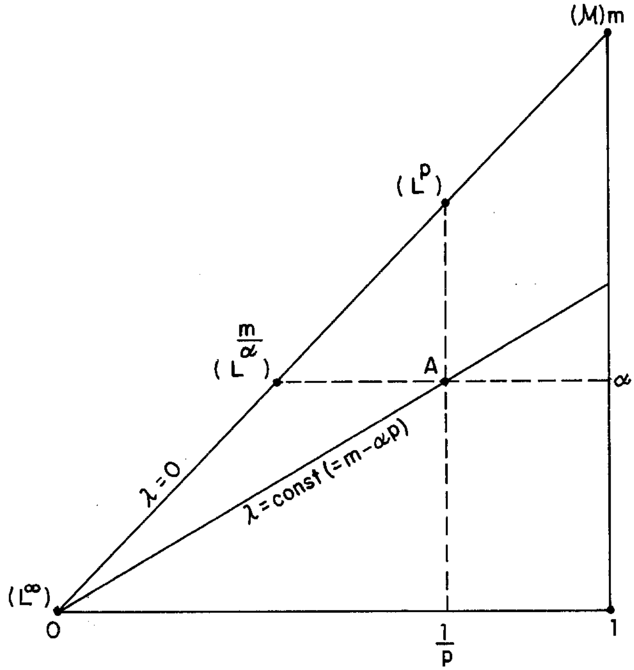

Figure 1

Also we use the obvious notations $\ddot{M}(A)$ and $\dot{M}(A)$. For $A = O$, we define $\dot{M}(O) \subset M(O) = L^{\infty}$ as the set of bounded continuous functions, and $\ddot{M}(O)$ as the set of bounded, uniformly continuous functions. We note that $M(A) \cap M(O) \subset \dot{M}(A)$.

The points $A \in \blacktriangle$ corresponding to $\lambda =$ const are on a ray emanating from O. The hypotenuse of $\blacktriangle$ corresponds to $\lambda = 0$, and represents the spaces L^p or $\mathcal{M}$, according as $p > 1$ or $p = 1$. The vertical side $(x(A) = 1)$ of $\blacktriangle$ corresponds to spaces of measures and is singular in many respects. Finally, we write $A \subset B$ if $y(A) = y(B)$ and B is to the right of A.

With these notations, (1.2) may be expressed by

(I) (inclusion) $A \subset B$ implies that $M(A) \subset M(B)$, $\dot{M}(A) \subset \dot{M}(B)$, and $\ddot{M}(A) \subset \ddot{M}(B)$.

It should be noted, however, that a relation like $M(A) \subset \dot{M}(B)$, suggested by analogy with Hölder spaces, is not true. For a counterexample, see Example 1.1, (b), below.

Other important properties of Morrey spaces are also conveniently expressed by the notation (1.3).

(II) (the Hölder inequality) $f \in M(A)$ and $g \in M(B)$ imply $fg \in M(A+B)$, with $\|fg; A+B\| \leq \|f;A\|\,\|g;B\|$. (Here A, B are regarded as 2-vectors.)

Proof. This follows from the Hölder inequality $\|fg\|_{r;x,R} \leq \|f\|_{p;x,R}\,\|g\|_{q;x,R}$ where $1/r = 1/p + 1/q$, combined with (1.4).

(III) (convexity) $f \in M(A) \cap M(B)$ implies $f \in M(C)$ for $C \in [AB]$, with $\|f;C\| \leq \|f;A\|^{1-k}\,\|f;B\|^{k}$, where $k = [AC]/[AB]$. ($[AB]$ may also denote the length of the segment $[AB]$, depending on the context.)

Proof. This follows easily from (II), combined with the simple identity

$$\|f^k; kA\| = \|f;A\|^k, \quad k > 0. \tag{1.5}$$

3. We add another kind of inclusion relation, using the original notation $M_{p,\lambda}$.

(IV) $M_{p,\lambda}$ is continuously embedded in the weighted L^p-space

$$L^p_{-k/p} = \langle x\rangle^{k/p}\, L^p \subset S'(=< x >^k \mathcal{M} \quad \text{if} \quad p=1)$$

for any $k > \lambda$, where $\langle x\rangle = (1+|x|^2)^{1/2}$.

Proof. With a continuous functions ϕ on $\mathbb{R}^+$, we have the identity

$$\int_{\mathbb{R}^m} \phi(|x|)\,|f(x)|^p\,dx = \int_0^\infty \phi(r)d\rho(r), \quad \rho(r) = \int_{|x|<r} |f(x)|^p\,dx. \tag{1.6}$$

If $f \in M_{p,\lambda}$, so that $\rho(r) \leq \|f\|^p\, r^\lambda$ ($\|f\| = \|f\|_{p,\lambda}$), and if $0 \leq \phi \in C^1$ with $\phi(r) = o(r^{-\lambda})$ for large r, an integration by parts gives

$$\int_{\mathbb{R}^m} \phi(|x|)\,|f(x)|^p\,dx \leq \int_0^\infty (-\phi'(r))\rho(r)dr \leq \|f\|^p \int_0^\infty |\phi'(r)|\, r^\lambda dr. \tag{1.7}$$

If we choose $\phi(r) =< r >^{-k}$ with $k > \lambda$, the integral converges, proving the required result. (If $p=1$, $|f(x)|\,dx$ should be interpreted as $|f|\,(dx)$.)

Finally we mention the Morrey spaces of vector valued functions. Given $f\colon \mathbb{R}^m \to \mathbb{R}^{m'}$, we say $f \in M(A)[\dot{M}(A), \ddot{M}(A)]$ if each component of f has the same property, and define the norm by $\|f;A\| = \|\,|f|\,;A\|$.

Example 1.1. (a) $L^p \subset \ddot{M}(A)$ if $p < \infty$ and $y(A) = m/p$.

(b) If $0 < k < m$, $|x|^{-k}$ belongs to $M(A)$ for $A = (1/p, k)$ with $1 \leq p < m/k$, but not to $\dot{M}(A)$ for any $A \in \blacktriangle$.

(c) Let $\nu = d\phi$, where ϕ is the Cantor ternary function defined on [0,1]. ν is a singular measure on [0,1]. Extend ν on all $\mathbb{R}$ by setting $\nu = 0$ outside [0,1].

Then $\nu \in M(1, \lambda)$ with $\lambda = \log(2)/\log(3)$. (This follows from the fact that ϕ is Hölder continuous with exponent λ.)

(d) If $\overline{m} < m$, $\overline{f} \in M(A)(\mathbb{R}^{\overline{m}})$ and if $f(x_1, \ldots, x_m) = \overline{f}(x_1, \ldots, x_{\overline{m}})$, then $f \in M(A)(\mathbb{R}^m)$. Thus the property $f \in M(A)$ remains unchanged when functions are "lifted" to higher dimensional space. (Note, however, that the triangle $\blacktriangle_m$ is larger for larger m.)

(e) The Dirac measure on $\mathbb{R}^{m-1}$ belongs to $M(1, m-1)$. Hence the "filament" in $\mathbb{R}^m$ concentrated on $x_1 = \ldots = x_{m-1} = 0$ belongs to $M(1, m-1)$ (apply (d) with $\overline{m} = m - 1$).

2. The heat operator

We now study the behavior of the heat operator $U(t) = e^{t\Delta}$ between Morrey spaces. Recall that $U(t)$ is a convolution operator with kernel

$$h_t(x) = \overline{h}_t(|x|), \quad \overline{h}_t(r) = ct^{-m/2}\exp(-r^2/4t), \quad t > 0. \tag{2.1}$$

(We denote various constants by c.) It is easy to see that $U(t)f = h_t * f$ is well defined for $f \in M(A)$ and belongs to $C^\infty(\mathbb{R}^m)$, with all the derivatives bounded.

Lemma 2.1. *Let $A, B \in \blacktriangle$, with B on or to the right of the segment $[OA]$. Let $\alpha = y(A)$, $\beta = y(B)$, so that $0 \leq \beta \leq \alpha \leq m$. If $t > 0$, the operators $U(t)$, $W(t) = \partial U(t)$, and $\partial_t U(t)$ are bounded from $M(A)$ to $\ddot{M}(B) \subset M(B)$, and depend on t continuously (in norm). Moreover, we have for $f \in M(A)$*

$$t^{\alpha/2-\beta/2}\,\|U(t)f; B\| \leq c\,\|f; A\|\,, \text{ with } c = 1 \text{ if } A = B, \tag{2.2}$$

$$t^{1/2+\alpha/2-\beta/2}\,\|W(t)f; B\| \leq c\,\|f; A\|\,; \tag{2.2a}$$

$$t^{1+\alpha/2-\beta/2}\,\|\partial_t U(t)f; B\| \leq c\,\|f; A\|\,. \tag{2.2b}$$

The constants c depend on A, B, α and β.

Proof. Set $f_t = U(t)f = h_t * f$, $t > 0$. The case $A = O$ admits only $B = O$, and the proof is easy. So we assume that $A \neq O$ in the sequel. Since $\|h_t\|_1 = 1$, the Hölder inequality gives

$$|f_t|^p \leq h_t * |f|^p\,, \quad 1 \leq p < \infty. \tag{2.3}$$

Using $\|h_t\|_1 = 1$ again, we obtain $\|f_t\|^p_{p;x,R} \leq \|f\|^p_{p,\lambda}\,R^\lambda$, which implies that $\|f_t\|_{p,\lambda} \leq \|f\|_{p,\lambda}$. With $\lambda = m - \alpha p$, this proves (2.2) for $B = A = (1/p, \alpha)$. Similarly, (2.2a) and (2.2b) for $B = A$ follow by using the inequalities

$$|\partial h_t(x)| \leq ct^{-1/2}\overline{h}_{2t}(|x|), \quad |\partial_t h_t(x)| \leq ct^{-1}\overline{h}_{2t}(|x|). \tag{2.4}$$

Next we prove (2.2) for $B = O$ (so that $M(B) = L^\infty$). (2.3) gives $|f_t(0)|^p \le \int \overline{h}_t(|x|)\,|f(x)|^p\,dx$. Hence we have by (1.7)

$$\begin{aligned} |f_t(0)|^p &\le \int_0^\infty \left|\overline{h}_t'(r)\right| \|f\|^p\, r^\lambda dr \\ &\le ct^{-1-m/2}\int_0^\infty \|f\|^p\, e^{-r^2/4t} r^{\lambda+1} dr \\ &\le c\,\|f\|^p\, t^{-(m-\lambda)/2}, \quad \|f\| = \|f\|_{p,\lambda}\,. \end{aligned} \tag{2.5}$$

Since $\alpha = (m-\lambda)/p$ for $M_{p,\lambda} = M(A)$, and since $x = 0$ is not a distinguished point, this proves (2.2) for $B = O$. Similarly, (2.2a) and (2.2b) are proved for $B = O$ by using (2.4).

Next we prove (2.2)-(2.2b) in the general case. In view of the inclusion property (I), we may assume that $B = (1/q, \beta) \in [OA]$, so that $\beta q = \alpha p$. Then

$$\|f_t\|_{q;x,R} \le \|f_t\|_\infty^{1-p/q}\,\|f_t\|_{p;x,R}^{p/q} \le \|f_t\|_\infty^{1-p/q}\,(\|f_t;A\|\,R^{m/p-\alpha})^{p/q}.$$

Since $\|f_t;A\| \le \|f;A\|$ as shown above, and since $(m/p-\alpha)(p/q) = m/q-\beta$, it follows that

$$\|f_t;B\| = \sup\{R^{-m/q+\beta}\,\|f\|_{q;x,R}\,; R > 0\} \le \|f_t;O\|^{1-p/q}\,\|f;A\|^{p/q}\,. \tag{2.6}$$

In view of (2.2) for the special case $B = O$ already proved, we thus obtain (2.2) (recall that $\alpha p = \beta q$). (2.2a) and (2.2b) follow in the same way from (2.4) and the special case $B = O$.

Finally, it remains to prove that $f_t \in \ddot{M}(B)$. This follows easily from (2.2a), which implies (1.1b) for $M_{p,\lambda} = M(B)$. Similarly, (2.2b) shows that $\partial_t U(t) \in B(M(A); M(B))$ is bounded locally uniformly for $t > 0$. Hence $U(t) \in B(M(A); M(B))$ is continuous in $t > 0$. The continuity of $W(t) = \partial U(t)$ can be proved similarly by estimating $\partial_t W(t)$ in the same way as above. The continuity of $\partial_t U(t) = \Delta U(t) = \partial U(t/2)\cdot\partial U(t/2)$ follows from that of $\partial U(t) = W(t)$, etc.

Lemma 2.2. *Let $f \in \dot{M}(A)$, $A \in \blacktriangle$. If $\alpha > \beta$, the left members of (2.2)-(2.2b) tend to zero as $t \to 0$. (Hence $t^{\alpha/2-\beta/2}U(t)$, $t^{1/2+\alpha/2-\beta/2}W(t)$, and $t^{1+\alpha/2-\beta/2}\partial_t U(t)$ are strongly continuous for $t \ge 0$ on $\dot{M}(A)$ to $\ddot{M}(B)$, with values zero at $t = 0$.)*

Proof. First we prove the assertion for (2.2) with $B = O$. According to the definition of $\dot{M}(A) = \dot{M}_{p,\lambda}$, we may replace $\|f\|$ in (2.5) by a function $F(r) \le \|f\|$ that tends to zero with r. Given any $\varepsilon > 0$, therefore, we may choose $\delta > 0$

in such a way that $F(r)^p < \varepsilon$ for $r < \delta$, obtaining

$$t^{(m-\lambda)/2} |f_t(0)|^p \leq ct^{-1-\lambda/2} \left(\varepsilon \int_0^\delta + \|f\|^p \int_\delta^\infty \right) e^{-r^2/4t} r^{\lambda+1} dr$$

$$= c \left(\varepsilon \int_0^{t^{-1/2}\delta} + \|f\|^p \int_{t^{-1/2}\delta}^\infty \right) e^{-s^2/4} s^{\lambda+1} ds.$$

On letting $t \to 0$ we see that limsup of the right member does not exceed $c\varepsilon$. Since ε was arbitrary, the left member must tend to zero with t. Since the same result holds when $f_t(0)$ is replaced by $f_t(x)$ for general $x \in \mathbb{R}^m$, uniformly in x, we have proved the lemma for (2.2) (recall that $m - \lambda = \alpha p$). (2.2a) and (2.2b) for $B = O$ can be handled in the same way.

That (2.2) tends to zero in the general case $\beta < \alpha$ can be seen from (2.6), in which $\|f_t; O\| = o(t^{-\alpha/2})$ by what was just proved (recall that we may assume $\alpha p = \beta q$). The same results for (2.2a) and (2.2b) follow similarly.

Remark 2.3. (2.2) and (2.2b) with $B = A$ show that $\{U(t)\}$ forms a bounded analytic semigroup on $M(A)$. As we shall see in next section, it is a C_0-semigroup on $\ddot{M}(A)$ but in general not on $M(A)$ or $\dot{M}(A)$. (This is natural since $U(t)M(A) \subset \ddot{M}(A)$.)

On the other hand, $a \in M_{p,\lambda}$ implies $a \in L^p_{-k/p}$ by (IV) (section 1) if $1 < p < \infty$ and $k > \lambda$. Since $U(t)$ is a C_0-semigroup on $L^p_{-k/p}$, we have $U(t)a \to a$, $t \to 0$, in the norm of $L^p_{-s/p}$. Of course this implies local L^p-convergence.

Lemma 2.4. *Let $A, B \in \blacktriangle$, with B on or to the right of $[OA]$. Let $\alpha = y(A)$, $\beta = y(B)$ (so that $\beta \leq \alpha$). If $t > 0$, we have for $f \in M(A) \cap M(B)$,*

$$\|W(t)f\|_\infty \leq ct^{-1/2-\beta/2}(1+t)^{\beta/2-\alpha/2} \|f; A \cap B\|, \tag{2.7}$$

$$\|W(t)f; B\| \leq ct^{-1/2}(1+t)^{\beta/2-\alpha/2} \|f; A \cap B\|, \tag{2.7a}$$

where $\|f; A \cap B\|$ is short for $\|f; M(A) \cap M(B)\|$.

Proof. Set $g(t) = W(t)f$. We have $\|g(t)\|_\infty \leq ct^{-1/2-\beta/2} \|f; B\|$ and $\|g(t)\|_\infty \leq ct^{-1/2-\alpha/2} \|f; A\|$, by (2.2a). Hence we obtain (2.7). (2.7a) can be proved in the same way.

3. The heat semigroup and the translation group

In what follows we denote by A a generic point of $\blacktriangle$, but occasionally use the old notation $M_{p,\lambda}$, if $A \neq O$.

We have shown (Lemma 2.1) that $U(t)M(A) \subset \ddot{M}(A)$. We shall now prove that $\ddot{M}(A)$ is in fact the maximal closed subspace of $M(A)$ on which $U(t)$ is a C_0-semigroup. This question is closely related to the continuity of the translation group $\{\tau_\xi; \xi \in \mathbb{R}^m\}$, since the heat semigroup is, like many convolution semigroups, *subordinated* to the translation group. We note the obvious fact that $M(A), \dot{M}(A)$ and $\ddot{M}(A)$ are translation invariant.

Lemma 3.1. *Let $f \in M(A)$. The following conditions are equivalent.*
(a) $f \in \ddot{M}(A)$.
(b) $\|\tau_\xi f - f; A\| \to 0$ *as* $\xi \to 0$.
(c) $\|U(t)f - f; A\| \to 0$ *as* $t \searrow 0$.

Corollary 3.2. *$\ddot{M}(A)$ is the maximal closed subspace of $M(A)$ on which the family τ_ξ forms a strongly continuous group and, at the same time, the maximal subspace on which $U(t)$ is a C_0-semigroup.*

Proof. For each $f \in M(A)$, we write $f_\xi = \tau_\xi f$, $f_t = U(t)f$.
(a) and (b) are equivalent, by definition.
(b) implies (c). Indeed, we have

$$f_t(x) - f(x) = \int h_t(\xi)[f(x-\xi) - f(x)]d\xi = \int h_t(\xi)[f_\xi(x) - f(x)]d\xi.$$

The last member may be interpreted as an integral of the $M(A)$-valued function $h_t(\xi)(f_\xi - f)$, which is continuous in ξ by hypothesis. Hence

$$\|f_t - f; A\| \leq \int h_t(\xi)\, \|f_\xi - f; A\|\, d\xi. \tag{3.1}$$

Since $\|f_\xi - f; A\|$ is bounded by $2\,\|f; A\|$ and tends to zero as $\xi \to 0$, a standard argument shows that (3.1) tends to zero with t. This proves (c).

Finally, (c) implies (a) because $f_t \in \ddot{M}(A)$ for $t > 0$, by Lemma 2.1.

Corollary 3.3. $\ddot{M}(A) \subset \dot{M}(A)$.

Proof. Let $f \in \ddot{M}(A)$. Then $f_t \to f$ in $M(A)$ as $t \to 0$, by Lemma 3.1. But $f_t \in \dot{M}(A)$, since $f_t \in M(A) \cap L^\infty$ by (2.2) with $B = O$. Hence $f \in \dot{M}(A)$.

Example 3.4. We give an example which shows that $\ddot{M}(A)$ is in general a proper subspace of $\dot{M}(A)$. Let us assume for simplicity that $m = 1$ and introduce the *wave packets*

$$\phi_n(x) = \sin(2\pi n x) \text{ for } 0 < x < 1, \quad = 0 \text{ elsewhere}, \quad n = 1, 2, \ldots .$$

If we set

$$f(x) = \phi_1(x - a_1) + \phi_2(x - a_2) + \dots, \qquad \text{where } a_{n+1} - a_n > 2,$$

then $f \in L^\infty$ with $\|f\|_\infty = 1$. This implies that $\|f\|_{p;x,R} = o(R^{\lambda/p})$ as $R \to 0$ for any $\lambda < 1$. Moreover, with any $\lambda \in (0,1)$ fixed, we can achieve that $f \in M_{p,\lambda}$ by letting a_n grow sufficiently fast. Thus $f \in \dot{M}_{p,\lambda}$ for $p \geq 1$.

On the other hand we have $\tau_{1/2n}\phi_n \approx -\phi_n$ for large n. Therefore

$$\left\|(\tau_{1/2n} - 1)f\right\|_{p;a_n+1/2,1} \approx 2,$$

with the error tending to zero as $n \to \infty$. Hence

$$\left\|(\tau_{1/2n} - 1)f\right\|_{p,\lambda} \geq 1 \quad \text{as} \quad n \to \infty,$$

showing that (1.1b) is violated. This implies that $f \notin \ddot{M}_{p,\lambda}$.

4. Convolution operators

We prove some results on convolution operators on Morrey spaces. Most of the following results are found in [6], but we shall give elementary proofs.

Lemma 4.1. *Let $S: \mathbb{R}^m \to \mathbb{R}$ such that*

$$|S(x)| \leq c|x|^{\delta - m}, \text{ where } 0 < \delta < m. \tag{4.1}$$

Let $A = (1/p, \alpha)$, $B = (1/q, \beta)$, with

$$0 < \beta < \alpha < m, \ \alpha - \beta = \delta, \ m/p - m/q \leq \delta \ (< \delta \text{ if } p = 1). \tag{4.2}$$

Then $S*$ (convolution with S) is bounded from $M(A)$ $[\dot{M}(A), \ddot{M}(A)]$ to $M(B)$ $[\dot{M}(B), \ddot{M}(B)]$.

Proof. 1. In view of the inclusion property (I), we may assume that

$$m/p - m/q = \delta \text{ if } p > 1; \quad m/p - m/q < \delta \text{ if } p = 1. \tag{4.3}$$

For each $\rho > 0$, let $S_\rho(x) = S(x)$ for $|x| < \rho$ and $= 0$ otherwise. Let $f \in M(A)$. We shall estimate $g' = S_\rho * f$ and $g'' = (S - S_\rho) * f$ separately.

First we show that $g'' \in L^\infty$. Let $1/p' = 1 - 1/p$ and choose positive numbers r, s such that

$$r/p + s/p' = m - \delta, \quad r > m - \alpha p, \quad s > m; \tag{4.4}$$

this is possible since $(m-\alpha p)/p + m/p' = m - a < m - \delta$. Then

$$|g''(x)| \le c \left(\int_{|y|>\rho} |y|^{-s}\, dy \right)^{1/p'} \left(\int_{|y|>\rho} |y|^{-r} |f(x-y)|^p\, dy \right)^{1/p}.$$

The first factor is equal to $c\rho^{(m-s)/p'}$, while the second one is majorized by $c\rho^{(\lambda-r)/p} \|f; A\|$ as in (1.7), where $\lambda = m - \alpha p$. It follows that

$$|g''(x)| \le c\rho^{(m-s)/p' + (m-\alpha p - r)/p} \|f; A\| \le c\rho^{-\beta} \|f; A\|,$$

where we have used (4.4). Thus $g'' \in L^\infty$ and, consequently,

$$\|g''\|_{q;x,R} \le cR^{m/q} \rho^{-\beta} \|f; A\|. \tag{4.5}$$

2. Next we show that

$$\|g'\|_{q;x,R} \le c(R+\rho)^{m/q-\beta} \|f; A\|. \tag{4.6}$$

For this we note that the values of f outside the ball $B_{R+\rho}(x)$ have no contribution to the left member. Hence we have, on setting $\overline{f} = f$ on $B_{R+\rho}(x)$ and $= 0$ elsewhere,

$$\|g'\|_{q;x,R} = \left\| S_\rho * \overline{f} \right\|_{q;x,R} \le \left\| S_\rho * \overline{f} \right\|_q. \tag{4.7}$$

If $p > 1$, this does not exceed

$$c \left\| \overline{f} \right\|_p \le c \|f\|_{p;x,R+\rho} \le c(R+\rho)^{m/p-\alpha} \|f; A\|$$

by the Sobolev inequality. Since $m/p - \alpha = m/q - \beta$ by (4.3), this proves (4.6).

If $p = 1$, (4.7) does not exceed $\|S_\rho\|_q \|\overline{f}\|_1$, where $\|S_\rho\|_q \le c\rho^{\delta - m + m/q}$ (note that $\delta - m + m/q > 0$ by (4.2)), while $\|\overline{f}\|_1 \le (R+\rho)^{m-\alpha} \|f; A\|$. Since $\delta - m + m/q + m - \alpha = m/q - \beta$, we have proved (4.6).

Since $S * f = g' + g''$, (4.5) and (4.6) give

$$\|S * f\|_{q;x,R} \le (cR^{m/q}\rho^{-\beta} + c(R+\rho)^{m/q-\beta}) \|f; A\|. \tag{4.8}$$

Given any $R > 0$, we may choose $\rho = R$. Then (4.8) gives $\|S * f\|_{q;x,R} \le cR^{m/q-\beta} \|f; A\|$. This implies that $S * f \in M(B)$ with $\|S * f; B\| \le c\|f; A\|$, as required.

3. Next assume that $f \in \dot{M}(A)$. For any $\varepsilon \in (0,1)$, there is $R_\varepsilon > 0$ such that $\|f\|_{p;x,R+\rho} \le \varepsilon^{m/\beta q}(R+\rho)^{m/p-\alpha} \|f; A\|$ for any $x \in \mathbb{R}^m$ if $R + \rho < 2R_\varepsilon$. This leads to an analog of (4.8):

$$\begin{aligned} &\|S * f\|_{q;x,R} \le (cR^{m/q}\rho^{-\beta} + c\varepsilon^{m/\beta q}(R+\rho)^{m/q-\beta}) \|f; A\|, \\ &R + \rho < 2R_\varepsilon. \end{aligned} \tag{4.9}$$

Now let $R < \varepsilon^{1/\beta} R_\varepsilon < R_\varepsilon$, and set $\rho = R\varepsilon^{-1/\beta} < R_\varepsilon$. Then $R + \varepsilon < 2R_\varepsilon$ and (4.9) gives, after a simple computation,

$$\|S * f\|_{q;x,R} \leq c\varepsilon R^{m/q-\beta} \text{ if } R < \varepsilon^{1/\beta} R_\varepsilon.$$

This shows that $S * f \in \dot{M}(B)$.

Finally, that S^* maps $\ddot{M}(A)$ into $\ddot{M}(B)$ follows from Lemma 3.1, (b), and the fact that S^* commutes with translations.

Lemma 4.2. *Let $K: \mathbb{R}^m \setminus \{0\}: \mathbb{R}$ be a singular kernel of Calderón-Zygmund type, i.e. a homogeneous continuous function of degree $-m$ with integral zero on any sphere about the origin. Let $A \in \mathcal{A}$, $0 < x(A) < 1$. Then $K*$ is bounded on $M(A)$ $[\dot{M}(A), \ddot{M}(A)]$ into itself.*

Proof. It suffices to modify the proof of Lemma 4.1 slightly, on setting $q = p$, $\delta = 0$, $\beta = \alpha$. With obvious notation, the same argument as above gives (4.5) for $g'' = (K - K_\rho) * f$, $q = p$, $\beta = \alpha$. On the other hand, (4.7) is true (with S_ρ replaced by K_ρ), with $q = p$. Since it is known that $K_\rho *$ is a bounded operator on L^p, $1 < p < \infty$, with bound independent of ρ, we have

$$\begin{aligned} \left\|K_\rho * \overline{f}\right\|_{p;x,R} &\leq \left\|K_\rho * \overline{f}\right\|_p \leq c\left\|\overline{f}\right\|_p \\ &= c\|f\|_{p;x,R+\rho} \leq c(R+\rho)^{m/p-\alpha}\|f;A\|. \end{aligned}$$

The remaining arguments are the same as in Lemma 4.1.

Lemma 4.3. *Lemmas 2.1, 2.2, and 2.4 remain true when $U(t)$ is replaced by $U(t)\Pi$, with possible change of the constants c, except for the cases $A = B = O$ and $x(A) = x(B) = 1$ (if applicable).*

Proof. Π is a special case of the $K*$ of Lemma 4.2. The assumption excludes the case $A = O$ in those lemmas. First consider Lemmas 2.1 and 2.4. The results are obvious if $x(A) < 1$, since Π is then bounded on $M(A)$ by Lemma 4.2. Suppose that $x(A) = 1$. If $0 < x(B) < 1$, then Π is bounded on $M(B)$, so that $U(t)\Pi = \Pi U(t)$, etc. satisfy the same estimates as $U(t)$, etc. If $x(B) = 1$ but B is lower than A, then $\Pi U(t)$, etc. are bounded from $M(A)$ to $M(B')$, where $B' \in [OA]$ with $y(B') = y(B)$. Hence the results follow by property (I). If $B = O$, we may use a factorization such as $U(t)\Pi = U(t/2)\Pi U(t/2)$ and let it act from $M(A)$ to $M(A/2)$ to $M(O) = L^\infty$, with Π acting in $M(A/2)$ where it is bounded.

To prove the result for Lemma 2.2, we may use the factorization

$$t^{\alpha/2-\beta/2}U(t)\Pi = (t^{\alpha/4-\beta/4}U(t/2))\Pi(t^{\alpha/4-\beta/4}U(t/2)), \quad \text{etc.}$$

acting from $M(A)$ to $M(C)$ to $M(B)$, where $C = A/2 + B/2$. The rightmost factor is strongly continuous on $\dot{M}(A)$ to $\ddot{M}(C)$ in $t \geq 0$, with value zero at $t = 0$, by Lemma 2.2. The remaining factor, which acts from $\ddot{M}(C)$ to $\ddot{M}(B)$, is bounded uniformly in t. Hence we obtain the required result.

Remark 4.4. If $p > 1$ and $\lambda < k < m$, Π is bounded also on $L^p_{-k/p} = \langle x\rangle^{k/p}L^p \supset M_{p,\lambda}$ (see (IV), section 1), due to the boundedness of singular integral operators in weighted L^p-spaces (see Coifman-Fefferman [2], Stein [8]). Π is not bounded on $M_{1,\lambda}$, or on $\langle x\rangle^k\mathcal{M} \supset M_{1,\lambda}$, $k > \lambda$. Spaces which contain $\langle x\rangle^k\mathcal{M}$ and on which Π is bounded are found among weighted Sobolev spaces $\mathbb{K} = \mathbb{K}^{1+\varepsilon}_{-\eta,-k} \equiv (1-\Delta)^{\eta/2}L^{1+\varepsilon}_{-k}$, where $\varepsilon > 0$ and $\eta > m\varepsilon/(1+\varepsilon)$. $\mathbb{K}$ is a reflexive Banach space and contains $\langle x\rangle^k\mathcal{M}$, by the Sobolev embedding theorem. Moreover the inclusion is compact, since $\mathbb{K}^{1+\varepsilon}_{-\eta,-k}$ is compactly embedded in the same type of space with larger k and η (cf. Prosser [7]). Since Π commutes with $1-\Delta$, it follows by [2] that Π is bounded on $\mathbb{K}$ if $-m\varepsilon/(1+\varepsilon) < k < m/(1+\varepsilon)$. We also note that $U(t)$ and $t^{1/2}W(t)$ are bounded on $\mathbb{K}$, uniformly in $t \in (0, T']$, $T' < \infty$, and that $U(t) \to 1$ as $t \to 0$ strongly in $\mathbb{K}$.

5. The integral equation

In this section we solve the integral equation

$$(INT) \qquad \begin{aligned} u &= \Phi u \equiv u_0 + G(u,u), \\ G(u,v) &= -\int_0^t W(t-s)\cdot\Pi\cdot(u\otimes v)(s)ds, \end{aligned}$$

to which (NS) will be reduced (recall that $W(t) = \partial U(t)$). Here $u_0(t)$ is a given function such that $\partial \cdot u_0(t) = 0$. We shall solve (INT) in the closed subspaces $\hat{M}(A)$ of *divergence free vectors* in the vector- valued Morrey spaces $M(A)$, $A \in \mathbb{A}$. If $0 < x(A) < 1$, we may write $\hat{M}(A) = \Pi M(A)$, since Π is then a bounded projection in $M(A)$ by Lemma 4.2.

It will be seen that the boundedness of $t^{1/2-y(A)/2}u_0(t)$ in t in some $M(A)$ with $y(A) < 1$ is decisive for global solvability of (INT). To deal with such functions, it is convenient to introduce vector valued continuous functions with weighted sup-norm. Given a Banach space Z, we denote by $C_k(Z) = C_k((0,T); Z)$ the space of Z-valued continuous functions f on $(0,T)$ with the norm

$$||| f;Z |||_k = ||| f;Z |||_{k;0,T} = \sup_{0<t<T} t^k \|f(t);Z\| < \infty. \tag{5.1}$$

In particular $C_0 = BC$ (bounded continuous functions). In most cases we have $k \geq 0$, but $k < 0$ is not excluded. In $C_k((0,T);Z)$ we introduce the *seminorm*

$$\| f;Z \|_k = \limsup_{t\searrow 0} t^k \|f(t);Z\| = \lim_{\tau\searrow 0} ||| f;Z |||_{k;0,\tau}. \tag{5.2}$$

The subspace of C_k consisting of functions f with $\| f;Z \|_k = 0$ will be denoted by $\tilde{C}_k$.

In these definitions, $(0,T)$ may be replaced by a general interval (a,b); then t^k in (5.1-2) should be replaced by $(t-a)^k$, and $t \searrow 0$ by $t \searrow a$, etc.

We begin with a lemma estimating the nonlinear term $G(u,v)$.

Lemma 5.1. *Let $P,Q \in \blacktriangle$ such that $P+Q \in \blacktriangle$. Let*

$$u \in C_h((0,T);M(P)), \quad v \in C_k((0,T);M(Q)), \text{ where } h+k<1. \tag{5.3}$$

If $R \in \blacktriangle$ is on or to the right of the segment $[O,P+Q]$, with $0 \leq y(P)+y(Q)-y(R) \equiv \delta < 1$ (replace $\leq$ with $<$ if $x(P+Q)=1$), then we have

$$G(u,v) \in C_\ell((0,T);M(R)),$$

with

$$\begin{aligned} ||| G(u,v);R |||_\ell &\leq c||| u;P |||_h ||| v;Q |||_k, \\ \| G(u,v);R \|_\ell &\leq c \| u;P \|_h \| v;Q \|_k, \end{aligned} \tag{5.4}$$

where $\ell = h+k-(1-\delta)/2$. (c depends on $h,k,y(P),y(Q),y(R)$ but not on T.)

Proof. To prove the estimates in (5.4), we note that

$$\|G(u,v)(t);R\| \leq c\int_0^t (t-s)^{-(1+\delta)/2} \|(u\otimes v)(s);P+Q\|\, ds,$$

by Lemmas 2.1 and 4.3. Since $\|(u\otimes v)(s);P+Q\| \leq \|u(s);P\|\,\|v(s);Q\| \leq s^{-h-k}||| u;P |||_h ||| v;Q |||_k$, the required results follow. The continuity statement about $G(u,v)$ can be proved by using the following lemma, on setting

$$\mu = h+k, \quad \nu = (1+\delta)/2, \quad f(s,t) = s^\mu (t-s)^\nu W(t-s)\cdot\Pi\cdot(u\otimes v)(s).$$

Lemma 5.2. *Let Z be a Banach space, $f(s,t)$ a Z-valued continuous and bounded function defined for $0<s<t<T\leq\infty$. Let $\mu<1$, $\nu<1$, and set*

$$g(t) = t^{\mu+\nu-1}\int_0^t s^{-\mu}(t-s)^{-\nu} f(s,t)ds, \quad 0<t<T.$$

Then $g \in BC((0,T);Z)$. *If, in addition,* $\lim f(s,t) = \overline{f}$ *exists as* $s,t \to (0,0)$, *then* $g \in BC([0,T);Z)$ *with* $g(0) = \overline{f}B(\mu,\nu)$, *where* B *is the beta function.*

Proof. Introducing the new variable σ by $s = t\sigma$ gives

$$g(t) = \int_0^1 \sigma^{-\mu}(1-\sigma)^{-\nu} f(t\sigma,t)d\sigma.$$

g is bounded since f is. The continuity of g follows by dominated convergence theorem. The last statement of the lemma is obvious.

We now solve (INT) in Morrey spaces.

Theorem I. *Let* $A, 2A \in \blacktriangle$ *with* $0 < y(A) < 1$. *Let*

$$u_0 \in C_h((0,\infty); \hat{M}(A)), \quad h = 1/2 - y(A)/2. \tag{5.5}$$

Then there is $\delta > 0$ *such that if* $\| u_0; A \|_h < \delta$, *there is* $0 < T \leq \infty$ *such that (INT) has a unique solution* u *satisfying*

$$u \in C_h((0,T); \hat{M}(A)), \quad \| u; A \|_h < 2\delta. \tag{5.6}$$

Moreover,

$$u - u_0 \in C_{h'}((0,T); \hat{M}(A') \cap \ddot{M}(A')), \quad h' = 1/2 - y(A')/2, \tag{5.7}$$

for any $A' \in [O, 2A]$ *with* $2y(A) - 1 < y(A') \leq 2y(A)$ *(*$< 2y(A)$ *if* $x(2A) = 1$*). If* $u_0 \in \tilde{C}_h(\hat{M}(A))$, *the symbols* C *in* (5.6-7) *may be replaced by* $\tilde{C}$. *If the norm* $||| u_0; A |||_{h;0,\infty}$ *is sufficiently small, then we can set* $T = \infty$ *(global solution).*

Proof. First we construct a global solution when $||| u_0; A |||_{h;0,\infty} = K_0$ is small. Let E_K be the set of $f \in C_h((0,\infty); \hat{M}(A))$ with $|||f; A|||_h \leq K$. Application of Lemma 5.1 with $P = Q = R = A$, $h = k = 1/2 - y(A)/2$ (so that $\ell = h$) shows that $u \in E_K$ implies $G(u,u) \in C_h((0,\infty); \hat{M}(A))$ with $||| G(u,u); A |||_h \leq cK^2$; note that $(1 - \Pi)G(u,u) = 0$ is obvious. It follows that $\Phi u \in E_K$ if $K_0 + cK^2 \leq K$. This condition is met if $K_0 < 1/4c$, with $K = 2K_0/(1 + (1 - 4cK_0)^{1/2}) < 2K_0 < 1/2c$. Moreover, it is easy to see that Φ is a contraction on E_K in the metric induced by $|||\ \ ; A |||_h$ (cf.[5]). Since E_K is a complete metric space, it follows that Φ has a fixed point u. u is a global solution of (INT), which satisfies (5.6) with $T = \infty$ and $\delta = 1/4c$.

Next suppose that $\| u_0; A \|_h < 1/4c$. Then (5.2) shows that

$$||| u_0; A |||_{h;0,T} < 1/4c$$

if T is sufficiently small. Since the argument given above holds true when $(0,\infty)$ is replaced by $(0,T)$, we conclude that there is a solution u of (INT) on $(0,T)$ satisfying (5.6) with $\delta = 1/4c$.

Since $u - u_0 = G(u,u)$, (5.7) follows directly from Lemma 5.1: set $P = Q = A$, $R = A'$, $h = k = 1/2 - y(A)/2$, so that $\ell = 1/2 - y(A')/2$.

If $u_0 \in \tilde{C}_h(M(A))$, then $\| u_0; A \|_h = 0$ so that

$$\| u; A \|_h = \| G(u,u); A \|_h \le c \| u; A \|_h^2 \le cK \| u; A \|_h,$$

hence $\| u; A \|_h = 0$ (because $cK < 1$), showing that $u \in \tilde{C}_h(M(A))$ too. Another application of Lemma 5.1 then shows that $u \in \tilde{C}_{h'}(M(A'))$.

The uniqueness of the solution can be proved for sufficiently small t; here the contraction property can be used due to the assumption that $\| u; A \|_h < 1/2c$. To extend the result to possibly larger t, we may repeat the same argument starting with initial time $t = \sigma > 0$; here the proof is trivial since $u(\sigma) \in M(A)$, which implies that $u \in \tilde{C}_h((\sigma,T); M(A))$.

Remark 5.3. Suppose that u_0 satisfies, in addition to (5.5),

$$\partial u_0 \in C_k((0,\infty); \hat{M}(B)), \quad k = 1 - y(B)/2,$$

where B is on the ray extending $[OA[$, with $A + B \in ◭$, $1 - y(A) < y(B) < 2$. Then it can be shown that the solution u satisfies the same condition on $(0,T)$ and that $\partial(u - u_0)$ has properties analogous to (5.7).

6. Solution of (NS)

To begin with, it is necessary to clarify the meaning of a solution of (NS). Basically we take it in the sense of *classical ordinary differential equation* in t with values in $S'(\mathbb{R}^m)$. (In particular the initial value is taken in the S'-topology.) This interpretation coincides with the notion of the *weak solution* commonly used in evolution equations. Strong and weak solutions are distinguished only by the *class* of functions they are supposed to belong to.

Actually the solutions constructed will be smoother and satisfy the initial condition in a stronger topology, as described in the theorems given below. What is important is that the weak interpretation of (NS) is sufficient to ensure that (NS) is equivalent to (INT) with the special free term

$$u_0(t) = U(t)a, \tag{6.1}$$

if the integral in (INT) is also taken in the weak sense (at least initially). This follows easily from the fact that $U(t)$ is equicontinuous on S' for $0 \le t \le T < \infty$ and forms a C_0-semigroup with the (continuous) generator Δ.

Thus it suffices to apply the results of Theorem I with (6.1). To this end, it is convenient to introduce a *seminorm* in $M(A)$ by

$$| \phi; A | = \operatorname{dist}(\phi, \dot{M}(A)) \le \|\phi; A\|, \quad \phi \in M(A), \quad A \in \blacktriangle. \tag{6.2}$$

(This is equal to the norm of the equivalence class $\tilde{\phi}$, to which ϕ belongs modulo $\dot{M}(A)$.) The following lemma, which relates the two seminorms we have introduced, is a direct consequence of Lemmas 2.1-2 and the definition of $\dot{M}(A)$.

Lemma 6.1. *Let $A, B \in \blacktriangle$, $y(B) < y(A)$, with B on or to the right of $[OA]$. Then we have, for u_0 in* (6.1),

$$\| u_0; B \|_h \le c | a; A |, \quad h = y(A)/2 - y(B)/2. \tag{6.3}$$

We now state our main existence theorem for (NS). Here it is crucial that the initial value a is in $M(A_0)$ with $y(A_0) = 1$. For the spaces L^p_{-k} and $\mathbb{K}$, see (IV) in section 1 and Remarks 2.3, 4.4.

Theorem II. *Let $a \in \hat{M}(A_0)$, where $A_0 = (1/p, 1) \in \blacktriangle$. If the seminorm $| a; A_0 |$ is sufficiently small (which is the case if $a \in \dot{M}(A_0)$), there is $T > 0$ such that (NS) has a solution u satisfying*

$$u \in BC((0,T); \ddot{M}(A_0)) \cap BC([0,T); L^q_{-k}) \text{ for } 1 < q < p \text{ and } k > m - q$$
$$(q = p \text{ is allowed if } a \in \dot{M}(A_0)) \quad \text{if} \quad p > 1, \tag{6.4}$$

$$u \in BC([0,T); \mathbb{K}), \ \mathbb{K} = \mathbb{K}^{1+\varepsilon}_{-\eta,-k}, m - 1 < k < m \quad \text{if} \quad p = 1, \tag{6.4a}$$

$$u \in C_{h'}((0,T); \ddot{M}(A')) \text{ for } A' \in [O, A_0[, \quad h' = 1/2 - y(A')/2. \tag{6.5}$$

u is the unique solution within the class (6.5) with a small seminorm $\| u; A' \|$, for any particular A' with $0 < y(A') \le 1/2$. If, in particular, $p > 1$ and $a \in \ddot{M}(A_0)$, then (6.4) is strengthened to

$$u \in BC([0,T); \ddot{M}(A_0)). \tag{6.6}$$

If the norm $\|a; A_0\|$ is sufficiently small, then we may set $T = \infty$, i.e., u is a global solution and has the decay rate given by (6.5).

Remark 6.2. (a) In the original notation, $M(A_0) = M_{p,m-p}$, $M(A') = M_{p/y(A'),m-p}$, and $M(O) = L^\infty$.

(b) The theorems show how $u(0) = a$ holds in topologies stronger than that of S': in $M(A_0)$ for (6.6); in L^q_{-k} for (6.4), which implies local L^q convergence; and in $\mathbb{K}$ for (6.4a). The last one (for $p = 1$) is rather weak; it is expected that here we may take the weak* topology of measures, but we have no proof.

(c) Setting $A' = O$ in (6.5) gives $\|u(t)\|_\infty \leq ct^{-1/2}$.

(d) In view of the inclusion relation (I) (section 1), A_0 in (6.4) and A' in (6.5) may be replaced by any point to its right with the same height.

Proof of Theorem II. As remarked above, it suffices to apply Theorem I with u_0 given by (6.1). Since $a \in \hat{M}(A_0)$, (5.5) is satisfied for all $A \in [OA[$ by Lemma 2.1. Thus there is a large freedom of choice for A.

We choose $A = A_0/2$, so that $y(A) = 1/2$, and note that $||| \; u_0; A \; |||_{1/4}$ is small if $\|a; A_0\|$ is small, by Lemma 2.1. Similarly $\| \; u_0; A \; \|_{1/4}$ is small if $| \; a; A_0 \; |$ is small, by Lemma 6.1. Thus application of Theorem I shows that there is a solution u of (INT) with $u - u_0$ satisfying (5.7) for $A' \in]O, A_0[$. Since this is also true of u_0, it follows that u has the same property. Another application of Lemma 5.1 extends this range to $[O, A_0[$, thus proving (6.5).

The remainder of the proof is concerned with (6.4) and (6.4a), which describe the behavior of u in $M(A_0)$ or related spaces. To this end, we consider u_0 and $u - u_0$ separately.

First assume that $p > 1$. If $a \in \ddot{M}(A_0)$, then $u_0 \in C([0,T); M(A_0))$ by Corollary 3.2. Otherwise we have only $u_0 \in BC((0,T); M(A_0))$, by Lemma 2.1. On the other hand, we have $u_0 \in C([0,T); L^p_{-k})$, since $U(t)$ forms a C_0-semigroup on $L^p_{-k} \supset M(A_0)$ (see Remark 2.3).

If $a \in \dot{M}(A_0)$, then $u - u_0 \in \tilde{C}_0((0,T); M(A_0))$ by Theorem I; in this case $u - u_0 \to 0$, $t \to 0$, in $M(A_0)$. In the general case, (5.7) gives $u - u_0 \in C_{-\varepsilon/2}((0,T); M((1+\varepsilon)A_0))$ with a small $\varepsilon > 0$, which implies that $u - u_0 \to 0$ in $M((1+\varepsilon)A_0)$. Since (IV) implies that $M((1+\varepsilon)A_0) \subset L^q_{-k}$, where $q = p/(1+\varepsilon) < p$, we have $\left\|(u - u_0)(t); L^q_{-k}\right\| \to 0$. On the other hand, $a \in M(A_0) \subset M(1/q, 1)$ by (I), hence $u_0(t) \to a$ in L^q_{-h} with any $h > m - q$.

Summing up, we obtain (6.4) and (6.6) for $p > 1$.

If $p = 1$, we use the inclusion $M(A_0) \subset \mathbb{K} = \mathbb{K}^{1+\varepsilon}_{-\eta,-k}$. (Refer to Remark 4.4 for the following arguments, with k and ε chosen so that $\lambda < k < m/(1+\varepsilon)$.) As above we have $u_0(t) \to a$ in $\mathbb{K}$. On the other hand, $u - u_0 = G(u,u)$ satisfies

$$\| \; (u - u_0)(t); \mathbb{K} \; \| \leq \int_0^t (t-s)^{-1/2} \| \; u \otimes u(s); \mathbb{K} \; \| ds, \quad t \leq T' < \infty,$$

because $W(t-s)$ is bounded on $\mathbb{K}$ with bound $c(t-s)^{-1/2}$ and Π is bounded on $\mathbb{K}$. Since

$$\| u \otimes u(s); \mathbb{K} \| \le c \| u \otimes u(s); A_0 \| \le c \| u(s); A_0/2 \|^2 = O(s^{-1/2}) \quad (6.7)$$

by (6.5), it follows that $(u - u_0)(t)$ is bounded in $\mathbb{K}$ as $t \to 0$. We shall show that it actually tends to zero in $\mathbb{K}$.

To see this, we first note that it tends weakly to zero (see below for the proof). But $\mathbb{K}$ is compactly embedded in a similar space with slightly larger parameters k and η. Since the precise values of these parameters are irrelevant, we conclude that $(u - u_0)(t) \to 0$ strongly in $\mathbb{K}$. This proves that $u \in BC([0,T]); \mathbb{K})$ with $u(0) = a$. (The unboundedness of Π is the major source of difficulty in working with the space $M(1,1)$.)

The statement of the uniqueness in Theorem II follows from Theorem I.

We now prove the weak convergence used above. Since the boundedness in t is known, it suffices to show that $\langle (u - u_0)(t), \varphi \rangle \to 0$ for all $\varphi \in S$ (note that S is dense in $\mathbb{K}^*$). But we have

$$\langle (u - u_0(t), \varphi \rangle = \int \langle U(t-s)\Pi(u \otimes u)(s), -\partial\varphi \rangle ds. \quad (6.8)$$

An estimate analogous to the one deduced above then shows that (6.8) is $O(t^{1/2})$; note that ∂ in $W(t-s) = \partial U(t-s)$ has been moved to the right member in $\langle\ ,\ \rangle$. This proves the stated weak convergence.

Remark 6.3. (a) Suppose that $a \in S'$ such that the associated vorticity $\omega = \partial \wedge a$ is a measure in $M(1,2) = M_{1,m-2}$. Since $a = S * \omega$ with a potential operator $S*$ satisfying the condition of Lemma 4.1 with $\delta = 1$ (the Biot-Savart law), it follows that $a \in M(1/p, 1) = M_{p,m-p}$ for any $p < m/(m-1)$. Theorem II thus gives a global solution of (NS) if the seminorm $\| \omega; (1,2) \|$ is sufficiently small, since $\| S^*\omega; (1/p, 1) \| \le c \| \omega; (1,2) \|$ by Lemma 4.1. This recovers the results of [3] so far as the velocity u is concerned. For $m = 2$, it is known [4] that (NS) has a global solution for any (large) a with $\partial \wedge a \in \mathcal{M}$, a result not covered by our theorem. (We shall discuss the vorticity equation in section 9.)

(b) In Theorem II, the initial velocity field may be a measure $a \in M(1,1) = M_{1,m-1}$. For example, a may be a divergence free, tangential vector measure concentrated on a smooth $(m-1)$-dimensional manifold $\Sigma \subset \mathbb{R}^m$, if it is sufficiently weak. More precisely, let Σ be the image of $\mathbb{R}^{m-1}$ under a diffeomorphism γ, and let b be a divergence free vector field on $\mathbb{R}^{m-1}$. We may define a as the distribution given by

$$\langle a, \varphi \rangle = \int_{\mathbb{R}^{m-1}} b_k(y)\gamma_{jk}(y)\varphi_j(\gamma(y))dy \quad \text{(for vector-valued test functions } \varphi\text{)},$$

where $\gamma_{jk}(y) = \partial\gamma_j(y)/\partial y_k$. Then a is a measure supported on Σ, $a \in M_{1,m-1}$, and it is easy to see that $\partial \cdot a = 0$. Such a velocity field is supposed to be infinitely fast but infinitely thin, with finite flux. (If it is a uniform flow on a hyperplane, the problem is explicitly solvable.)

(c) In this paper we consider only free flows. Equation (NS) with a forcing term $f(t, x)$ on the right can be handled in the same way. The only modification needed is to replace $u_0(t)$ with

$$u_0(t) = U(t)a + \int_0^t U(t-s)f(s, \,\cdot\,)ds.$$

7. Decay

The global solutions given by Theorems II for small a decay for large t according to (6.5). In particular we have $\|u(t)\|_\infty = O(t^{-1/2})$. This rate could not be improved without further assumptions. It is well known that the decay is faster if a has faster spatial decay. A typical case is that $a \in M(1,m) = M_{1,0} = \mathcal{M}$ (finite measures); then the decay rate will be $O(t^{-m/2})$.

In this section we deal with continuous Z-valued functions on $(0,\infty)$ which are $O(t^{-h})$ as $t \to 0$ and $O(t^{-k})$ as $t \to \infty$. Such a class will be denoted by $C_{h,k}(Z)$. We write $C_{h,k} = C_{h,k}(\mathbb{R})$. The associated norm may be defined by (cf. (5.1))

$$|||f; Z|||_{h,k} = \sup_{0<t<\infty} t^h(1+t)^{k-h} \|f(t); Z\| .$$

The following rules of calculus with these classes are easy to verify.

$$f \in C_{h,k}(Z) \text{ if and only if } \|f(\,\cdot\,); Z\| \in C_{h,k}, \text{ with the same norm,} \tag{7.1}$$

$$C_{h,k}(Z) \subset C_{h',k'}(Z) \text{ if } h \le h' \text{ and } k \ge k', \tag{7.2}$$

$$C_{h,k} \cdot C_{h',k'} \subset C_{h+h',k+k'}, \tag{7.3}$$

$$C_{h,k} * C_{h',k'} \subset C_{h+h'-1,\ell} \quad \text{if} \quad \max\{h, h'\} < 1, \tag{7.4}$$

where $*$ denotes convolution on the half-line $(0,\infty)$, and where $\ell = \min\{k, k'\}$ if $\max\{k, k'\} > 1$, $\ell = k + k' - 1$ if $\max\{k, k'\} < 1$, and $\ell = k + k' - 1 - \varepsilon$ for any $\varepsilon > 0$ if $\max\{k, k'\} = 1$.

The main result of this section is given by

Theorem III. *Let $a \in \hat{M}(A_0) \cap M(B_0)$, where $A_0 = (1/p, 1) \in \blacktriangle$, $B_0 = (1/q, \beta) \in \blacktriangle$, $1 < \beta \leq m$, with the case $p = q = 1$ excluded. If $\|a; A_0\| + \|a; B_0\|$ is sufficiently small, the global solution u given by Theorem II satisfies $\|u(t)\|_\infty = O(t^{-\beta/2})$ as $t \to \infty$. Moreover, $u \in BC((0,\infty); M(B_0))$ if $q > 1$. (Note that $M(A_0) = M_{p,\lambda}$, $M(B_0) = M_{q,\mu}$, with $\lambda = m - p$, $\mu = m - \beta q$. The largest β is m, which occurs for $q = 1$, $\mu = 0$.)*

For the proof, we solve (INT) once more in a different class of functions than before. If a is sufficiently small, the solution obtained should be identical with that in Theorem II, by the uniqueness result.

In $\blacktriangle$, B_0 is higher than A_0 because $\beta > 1$, and the open segment $]B_0A_0[$ does not meet the line $x = 1$, since $p = q = 1$ is excluded. We assume, without loss of generality, that A_0 is on or to the right of the segment $[OB_0]$; otherwise we can move A_0 horizontally toward the right without affecting the assumption that $a \in M(A_0)$ (see property (I)). We introduce two more points A and Q. $A = (1/2p, 1/2)$ is the middle point of $[OA_0]$. Q is on the open segment $]A_0B_0[$, below B_0 vertically by an amount ε, where $0 < \varepsilon < \min\{\beta - 1, 1/2\}$ (see Fig. 2).

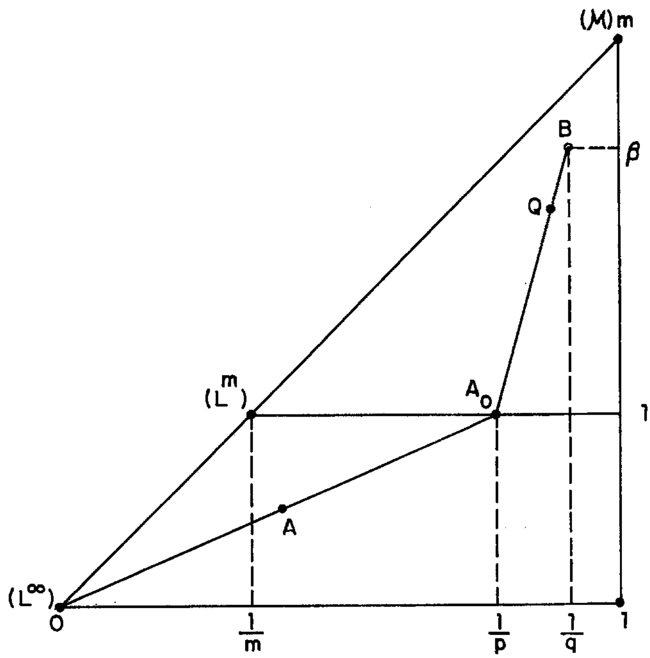

Figure 2

Lemma 7.1. *The operator valued function $W(t)\Pi = \partial e^{t\Delta}\Pi$ satisfies*

$$\begin{aligned} W\Pi \in\; & C_{3/4,1/2+(\beta-\varepsilon)/2}(B(M(A)\cap M(Q);M(O))) \\ & \cap C_{1/2,1/4+(\beta-\varepsilon)/2}(B(M(A)\cap M(Q);M(A))) \\ & \cap C_{1/2,1/2}(B(M(Q);M(Q))). \end{aligned} \tag{7.5}$$

Proof. Since $x(A) < 1$, $x(Q) < 1$, we may disregard the factor Π (cf. proof of Lemma 4.3). Set $g(t) = W(t)\phi$. We have $\|g(t);A\| \le ct^{-1/2}\|\phi;A\|$ and $\|g(t);A\| \le ct^{-1/4-(\beta-\varepsilon)/2}\|\phi;Q\|$, by Lemma 2.1; note that A is to the right of $[OQ]$, $y(A) = 1/2$, and that $\beta-\varepsilon > 1$. Hence we obtain the second part of (7.5). The first part can be proved in the same way. The last one is trivial.

We now set $Y = \hat{M}(0)\cap M(A)\cap M(Q)$ and define $E_K \subset C((0,\infty);Y)$ as the set of u satisfying the conditions

$$u \in C_{1/2,\beta/2}(M(0))\cap C_{1/4,\beta/2-1/4}(M(A))\cap C_{0,\varepsilon/2}(M(Q)), \tag{7.6}$$

with the associated norm $\le K$. We note that $u_0 = U(t)a$ satisfies (7.6) with norm $K_0 \le c(\|a;A_0\| + \|a;B_0\|)$, as is seen from Lemma 2.1; note that $a \in M(Q)$ by property (III). We choose $K > K_0$, so that $u_0 \in E_K$.

Lemma 7.2. *If $u \in E_K$, then*

$$u\otimes u \in C_{1/2,(\beta+\varepsilon)/2}(M(Q))\cap C_{3/4,(\beta+\varepsilon)/2}(M(A)\cap M(Q)), \tag{7.7}$$

with norm $\le cK^2$.

Proof. Since $\|u\otimes u(t);Q\| \le \|u(t);O\|\,\|u(t);Q\|$, (7.6) implies the first part of (7.7), by (7.3). Similarly, we obtain $u\otimes u \in C_{3/4,\beta-1/4}(M(A))$ from

$$\|u\otimes u(t);A\| \le \|u(t);O\|\,\|u(t);A\|.$$

These together imply the second part of (7.7) by (7.2); note that $\beta - 1/4 > (\beta+\varepsilon)/2$ because $\beta-\varepsilon > 1$.

Lemma 7.3. *If $u \in E_K$, then $G(u,u) \in E_K$ with norm $\le cK^2$.*

Proof. We have

$$\|G(u,u)(t);O\| \le$$
$$\le \int_0^t \|W(t-s);B(M(A)\cap M(Q);M(0)))\|\,\|u\otimes u(s);M(A)\cap M(Q)\|\,ds,$$

which may be written in a short-hand notation

$$\|G(u,u);O\| \le \|W;A\cap Q\to O\| * \|u\otimes u;A\cap Q\|.$$

Similarly we have

$$\|G(u,u);A\| \leq \|W;A\cap Q \to A\| * \|u\otimes u;A\cap Q\|,$$
$$\|G(u,u);Q\| \leq \|W;Q \to Q\| * \|u\otimes u;Q\|.$$

In view of (7.5), (7.7) and the rule (7.1), therefore, it suffices to prove that

$$C_{3/4,1/2+(\beta-\varepsilon)/2} * C_{3/4,(\beta+\varepsilon)/2} \subset C_{1/2,\beta/2},$$
$$C_{1/2,1/4+(\beta-\varepsilon)/2} * C_{3/4,(\beta+\varepsilon)/2} \subset C_{1/4,\beta/2-1/4},$$
$$C_{1/2,1/2} * C_{1/2,(\beta+\varepsilon)/2} \subset C_{0,\varepsilon/2}.$$

But these follow from (7.4); recall that $\beta - \varepsilon > 1$, $0 < \varepsilon < 1/2$. (We note that the proof works even for $\varepsilon = 0$ if $x(B_0) < 1$.)

Completion of the proof of Theorem III. The remaining arguments are similar to those given in section 5. We prove that Φ is a contraction map of E_K into itself. The fixed point u of Φ solves (INT), and hence (NS); it satisfies the required decay rate because $u \in E_K$. The identity of u with the solution given by Theorem II follows from the uniqueness result, since (7.6) implies that $\|u(t);A\| \leq Kt^{-1/4}$.

To prove the last assertion in Theorem III, we note that if $q > 1$ we can take $Q = B_0$ (so that $\varepsilon = 0$) in the proof above, and (7.6) shows that $\|u(t);B_0\| \leq K$.

Remark 7.4. Theorem III gives the decay only for the L^∞-norm. Other L^r-norms need not exist in general. If $q > 1$ and $\mu = 0$, however, then $M(B_0) = L^q$ and we have the decay

$$\|u(t)\|_r \leq Kt^{\gamma/2-\beta/2} = Kt^{m/2r-m/2q}, \quad t \to \infty, \tag{7.8}$$

for $q \leq r \leq \infty$, $\gamma = m/r$. Indeed, Theorem III shows that (7.8) is true for $r = \infty$ and $r = q$, hence for all r in between. We do not know if (7.8) holds when $q = 1$. But it does if $a \in L^1 \cap L^m$ (which corresponds to the case $p = m$, $q = 1$, $\beta = m$, $\mu = 0$ in Theorem III), at least for $1 < r \leq \infty$. In this case B_0, A_0, Q, A are all on the hypotenuse of $\blacktriangle$. If we choose $\varepsilon = 1/2$, we see from (7.6) that (7.8) holds for $2m/(2m-1) \leq r \leq \infty$. Using the fact that $B_0 = Q + A$ (which is special in this case), we can then use Lemma 5.1 to extend this range to $1 < r \leq \infty$.

8. Regularity

Theorem II does not exhibit much regularity of the solutions u of (NS). Actually they are smooth for $t > 0$. More precisely, we have

Theorem IV. *Let u be the solution given by Theorem II. Then*

$$\partial_t^k \partial^n u \in C((0,T); \ddot{M}(A')) \text{ for } A' \in [OA_0], \quad k,n = 0,1,2,\dots, \tag{8.1}$$

with $A' = A_0$ excluded if $p = 1$. Moreover, (8.1) is true if A' is replaced by any $A'' \supset A'$ (see (I), section I). (∂^n denotes the generic space derivative of order n. We do not consider the degrees of singularities of $\partial_t^k \partial^n u$ at $t = 0$).

Proof. 1. First we prove (8.1) for $k = 0$ and for the special point $A' = A \equiv A_0/2$, by induction on n. As is shown by Theorem II, it is true for $n = 0$. Assuming it for $n \leq N - 1$, we shall prove the same for $n = N$. (In this process only the values $t > 0$ are involved; therefore we may take any $\sigma > 0$ and prove the results for $t > \sigma$.)

To this end we solve (INT) in another space of functions on (σ, T) with a small $T - \sigma > 0$. Since the method is similar to the previous ones, we may be brief but note that the initial value $a = u(\sigma)$ is known to be in $\hat{M}(A) \cap \ddot{M}(A)$. This will give us stronger results than in the case $\sigma = 0$.

For simplicity we write $u^n = \partial^n u$. We define the set E_K of functions u such that

$$\begin{aligned} &u^n \in C([\sigma,T); \ddot{M}(A)) \text{ for } n = 0,1,\dots,N-1 \quad \text{and} \\ &u^N \in C_{1/2}((\sigma,T); \ddot{M}(A)), \end{aligned} \tag{8.2}$$

with norms not exceeding K, with T and K yet to be determined. Note that $u_0 = U(t)a \in E_K$ if K is chosen appropriately, since $a^n \in \ddot{M}(A)$ for $n = 0,1,\dots,N-1$ by induction hypothesis (see Lemma 2.1 again).

As before, we have to show that $u \in E_K$ implies $G(u,u) \in E_K$. A straightforward computation gives

$$\partial^n G(u,u) = G_{n,0}(u,u^n) + G_{n,1}(u^1,u^{n-1}) + \dots + G_{n,n}(u^n,u), \tag{8.3}$$

where the $G_{n,j}$ are bilinear integral operators on functions on (σ,T), with the same properties as $G(u,u)$. Using (8.2) and Lemma 5.1, it is easy to see that $G(u,u)$ satisfies (8.2); in fact $G(u,u)$ is better behaved in the sense that the C for $n \leq N - 1$ may be replaced by $C_{-1/4}$ and the $C_{1/2}$ for $n = N$ by $C_{1/4}$. This means that the norm of $G(u,u)$ in E_K has a factor $(T - \sigma)^{1/4}$, which is small with $T - \sigma$. By taking K as above and then choosing $T - \sigma$ small, we can thus achieve that Φ maps E_K into itself. Since it can be shown that Φ is also a contraction, we have a fixed point u of Φ in E_k, which is a solution of (INT). This u is identical with the original one, due to the uniqueness result in Theorem II, which is trivially applicable to the present case because $a \in \ddot{M}(A)$.

We have thus proved that u satisfies (8.2). Since (8.2) implies that $u^n \in C((\sigma,T);\hat{M}(A))$ for $n \leq N$, this completes the induction; recall that $\sigma > 0$ was arbitrary.

2. Next we extend the results to other A' than A. This is easily done by applying Lemma 5.1 to (8.3), again with initial time $t = \sigma > 0$. We thus obtain (8.1) for $A' \in]OA_0]$, and another application extends it to $A' \in [OA_0]$, completing the proof of (8.1) for $k = 0$. (If $p = 1$, exclude $A' = A_0$.)

3. Finally we prove (8.1) for $k \geq 1$, in two steps. First we prove (8.1) with $A' = O$ excluded, again by induction. Suppose that it has been proved for all the derivatives involving ∂_t no more than $N-1$ times. Differentiation of (NS) shows that a derivative involving ∂_t^N is the sum of a derivative of u and the images under Π of the products of two derivatives of u, all involving ∂_t no more than $N-1$ times. But these functions belong to all $C((0,T);M(A'))$ for $A' \in]O,A_0]$, by the induction hypothesis, and the same is true of their products by property (II). Since Π maps each $M(A')$ into itself, the induction is complete. (Here $A' = A_0$ is excluded if $p = 1$.)

To extend the result to include $A' = 0$, we use the integral expression. Applying a differential operator $D = \partial_t^k \partial^n$ to (NS) gives $\partial_t Du = \Delta Du + \Pi f$, where f is the sum of products of two space-time derivatives of u, so that f belongs to the class (8.1) with $A' = O$ excluded. Integration of this differential equation then gives

$$Du(t) = U(t-\sigma)Du(\sigma) + \int_\sigma^t U(t-s)\Pi f(s)ds$$

for any $\sigma > 0$. Another application of Lemmas 2.1 and 5.1 then shows that $Du(t) \in L^\infty = M(O)$ for $t > 0$.

9. Vorticity

The vorticity ς is a skew symmetric tensor given by

$$\varsigma = \partial \wedge u, \quad \text{or} \quad \varsigma_{ij} = \partial_i u_j - \partial_j u_i. \tag{9.1}$$

Since u constructed above is smooth for $t > 0$, the existence of ς is trivial; the main interest is in its behavior at $t = 0$ and $t = \infty$ (if applicable). Usually ς is constructed by solving the *vorticity equation*, which is obtained by taking the curl of (NS). But we shall rather regard the problem as a form of regularity theorem for (NS), which claims that $\partial \wedge u(t)$ exists in some Morrey spaces if $\partial \wedge a$ does.

A divergence free vector field $a \in S'$ is recovered from $\omega = \partial \wedge a$ by a formula $a = S * \omega$, where $S*$ is a potential operator of the form (4.1) with $\delta = 1$ (see Remark 6.3(a)). Thus it is natural to assume that $\omega \in M(V_0)$ with $y(V_0) = 2$, which implies $a = S * \omega \in \hat{M}(A_0)$ with $y(A_0) = 1$ and $x(A_0) \geq x(V_0) - 1/m$ (replace $\geq$ with $>$ if $x(V_0) = 1$), by Lemma 4.1. For simplicity, here we consider only the case that ω is a measure, so that $V_0 = (1, 2)$. (See Fig. 3).

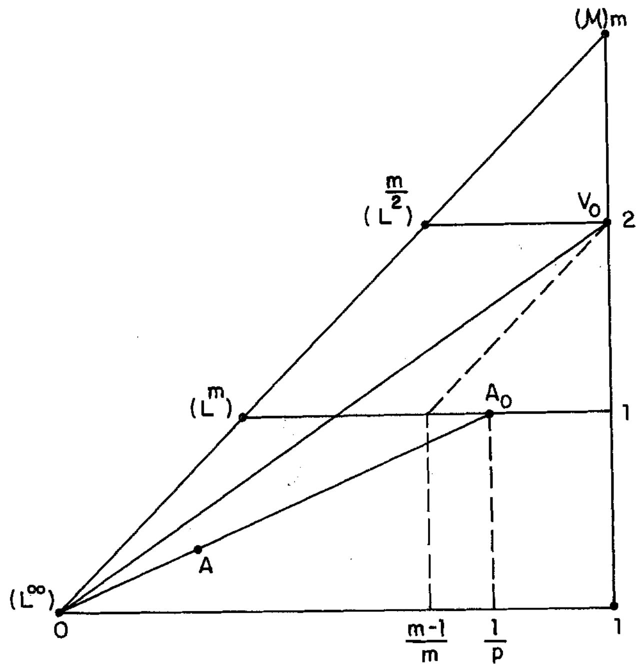

Figure 3

Theorem V. *Let* $a \in \hat{M}(A_0)$, $\partial \wedge a = \omega \in M(V_0)$ *(tensor valued), where* $V_0 = (1, 2)$ *and* $A_0 = (1/p, 1)$ *with* $1 < p < m/(m-1)$. *If the seminorm* $|\omega; V_0|$ *is sufficiently small, there is* $T > 0$ *and a unique solution* u *of (NS) satisfying* (6.4), (6.5) *and*

$$\begin{aligned} &\partial \wedge u \in BC((0,T);\ \ddot{M}(V_0)) \cap BC([0,T); \mathbb{K}), \\ &\mathbb{K} = \mathbb{K}^{1+\epsilon}_{-\eta,-k}, \quad for \quad k > m-2, \end{aligned} \tag{9.2}$$

$$\begin{aligned} &\partial \wedge u \in C_{k'}(0,T); \ddot{M}(B')) \quad for \quad B' \in]OV_0[, \\ &k' = 1 - y(B')/2. \end{aligned} \tag{9.3}$$

If the norm $\|\omega; V_0\|$ *is sufficiently small, we have a global solution* $(T = \infty)$, *which decays according to* (9.3).

Remark 9.1. Theorem V recovers the main results of [3] with measures as initial vorticity. (9.2) shows that $\varsigma = \partial \wedge u$ is better behaved compared with the velocity u when $u(0) = a$ is a measure, inasmuch as $\varsigma(t)$ stays (and is bounded) in $M(V_0)$. In particular, (9.2) implies that $\varsigma(0) = \omega$ holds in the weak* topology of measures (cf. Remark 6.2,(a)). On the other hand, we have no estimates for $\|\varsigma(t)\|_\infty$.

Proof. We solve (INT) by the familiar method using another space of functions. We set $A = A_0/3$, and define the set $E_{K,L}$ of functions u such that

$$\begin{aligned} &u \in C_{1/2}(\hat{M}(O)) \cap C_{1/3}(M(A)), \\ &\partial \wedge u \in C_0(M(V_0)) = BC(M(V_0)), \end{aligned} \tag{9.4}$$

where the interval $(0, T)$ is understood, with the norms of u and $\partial \wedge u$ majorized by K, L, respectively.

As before, we want to show that Φ maps $E_{K,L}$ into itself. By virtue of the assumptions and Lemma 2.1, the free term $u_0 = U(t)a$ satisfies (9.4) with certain norms K_0, L_0. For the term $G(u,u)$, the conditions on u in (9.4) are easily verified using Lemma 5.1; here we use $P = Q = A$, $R = O$ and then $P = Q = R = A$. To verify the condition on ∂u, we use the relation

$$\begin{aligned} \partial \wedge G(u,u)(t) &= G'(u, \partial \wedge u)(t) \\ &\equiv \int_0^t \partial \wedge U(t-s)(u \cdot \partial \wedge u)(s) ds, \end{aligned} \tag{9.5}$$

which follows by a simple algebra. Note that *the projection* Π *does not occur in* (9.5), since $\Pi - 1$, which maps into gradients, has been annihilated by $\partial \wedge$; otherwise G' is a bilinear operator similar to G. Application of Lemma 5.1 with $P = 0$, $Q = R = V_0$ then verifies the condition in (9.4) for $\partial \wedge G(u,u)$.

Thus the proof proceeds in the same way as in the foregoing theorems. After finding a fixed point of Φ, we use Lemma 5.1 again to cover the segment $]OV_0[$ in (9.3).

Here we may add the following remarks. First, due to the absence of Π in (9.5), the integral operator involved is bounded on $M(V_0)$ into itself, which makes it possible to apply Lemma 5.1 on $M(V_0)$ even though $x(V_0) = 1$. (This is responsible for the better behavior of ς stated in Remark 9.1.) Second, a small $\| \omega; V_0 \|$ [resp. $\|\omega; V_0\|$] implies a small $\| a; A_0 \|$ [resp. $\|a; A_0\|$] (see Remark 6.3,(a)).

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Tosio Kato
Department of Mathematics
University of California
Berkeley, CA 94720
USA